Composition operators on Sobolev spaces and Ball's classes

Vladimir Gol'dshtein
Ben-Gurion University of the Negev

August 29-September 3, Batumi, 2022
XII INTERNATIONAL CONFERENCE OF THE GEORGIAN MATHEMATICAL UNION

[0](Joint work with Alexander Ukhlov)

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary, $n \geq 3$. Let $\varphi : \overline{\Omega} \to \mathbb{R}^n$ be a mapping continuous on $\partial\Omega$ and one to one in Ω. Let $p > n$ and let $\psi \in W_q^1(\Omega)$ satisfy to $\psi|\partial\Omega = \varphi|\partial\Omega$, $detD(\psi) > 0$ a.e. and adj$(D\psi)$ belongs to $W_r^1(\Omega)$ for some $r > \frac{q}{q-1}$. This class was introduced by J. M. Ball in his fundamental papers (1976-1981) about nonlinear elasticity problems.

He demonstrated that solutions of such boundary valued nonlinear elasticity problems are minimizers of complicated energy integrals of deformations

$$I(\varphi,\Omega) := \int_{\Omega} W(x,\varphi(x),D\varphi(x))\ dx,$$

where $\varphi : \Omega \to \mathbb{R}^n$ is a continuous weak differentiable mapping (deformation) and these minimizers are homeomorphisms. The detailed discussion about properties of these variational problems can be found in (J.M. Ball, Some open problems in elasticity. In Geometry, Mechanics, and Dynamics, pages 3–59,. Springer, New York, 2002, 269 pages).
In this work we discuss an interplay between Ball's classes and composition operators on Sobolev spaces

In the spatial case ($n \geq 3$) J. M. Ball demonstrated that appropriate classes of minimizers for $q > n$ (so-called Ball's classes) belongs to

$$A^+_{q,r}(\Omega) = \left\{\varphi \in W^1_q(\Omega) : \operatorname{adj} D\varphi \in L_r(\Omega),\ J(x,\varphi) > 0 \text{ a. e. } x \in \Omega\right\},$$

In consequent works these classes were studied for $q > n - 1$ and $r \geq q/(q-1)$. J. M. Ball proved invertibility of deformations for $q > n$ in the case of incompressible bodies ($J(x,\varphi) \equiv 1$ and weak regularity of deformations). Namely φ^{-1} belongs to the class $W^1_1(\widetilde{\Omega})$ where $\widetilde{\Omega} = \varphi(\Omega)$.

We use the following preliminary information. By J. M. Ball minimizers are homeomorphisms for $p > n$. The case $p = n$ follows from (Vodop'janov. G, Siberian Math.J., 1979).
In the V.Sverak work (88) was proved invertibility of Ball's classes for $(n - 1 < q < n)$. This case is more complicated. V.Sverak also proved that Ball's mapping are homeomorphisms outside of so-called "singular" sets S of (n-q)-Hausdorff measure zero and the inverse mapping is continuous outside a set of (n-1)-dimensional Hausdorff measure zero. The Sverak's proof contains two gaps. He did not put attention that his change of variables formula is correct only if the mapping φ posses the Luzin N-property. He proved only that for any discontinuity point $x \in S$ limiting values of φ has (n-1)-dimensional Hausdorff measure zero.

The second gap was corrected by P. Hajlasz and P.Koskela (2003). They proved that q-dimensional capacity of the singular set S is zero and $(n-1)$-dimensional Hausdorff measure of its "image" is also zero. It means that after the deformation the body Ω has no cracks. We do not know any results about $p \leq n-1$. This restriction can clarified by capacitary estimates even in the case of continuity of inverse mapping.
Let us return to the definition of Ball's classes.

$$A^{+}_{q,r}(\Omega) = \left\{\varphi \in L^1_q(\Omega) : \operatorname{adj} D\varphi \in L_r(\Omega),\ J(x,\varphi) > 0 \text{ a. e. } x \in \Omega\right\},$$

This definition a little bit corrected. We changed W^1_q to L^1_q because the seminorm in this space corresponds to energy integrals.

The homogeneous seminormed Sobolev space $L^1_q(\Omega)$, $1 \leq q \leq \infty$, is defined as a space of locally integrable weakly differentiable functions $f : \Omega \to \mathbb{R}$ equipped with the following seminorm:

$$\|f \mid L^1_q(\Omega)\| = \|\nabla f \mid L_q(\Omega)\|.$$

Recall that by the Sobolev-Poincare inequality in Lipschitz domains $\Omega \subset \mathbb{R}^n$, $n \geq 2$, Sobolev spaces $W^1_p(\Omega)$ and $L^1_p(\Omega)$ are coincide (see, for example, (Maz'ya V.G. Sobolev Spaces). 2010.)
Let us remark that the seminorm in $L^1_q(\Omega)$ in degree q is the p-energy integral

$$\int_\Omega |\nabla f(x)|^q dx$$

. This slight modification corresponds to the original variational problem.

In the present work, using the geometric theory of composition operators on Sobolev spaces L^1_q we introduce a characterization of Ball's classes in terms of the composition operators on Sobolev spaces L^1_q. It permits us to obtain accurate volume distortion estimates of topological mappings (homeomorphisms) of Ball's classes possessing the Luzin N-property (estimates possible degree of deformation). The absolute continuity of the of Ball's classes with respect to the Sobolev capacity is proved also. The capacity considered as an outer measure associated with corresponding Sobolev spaces L^1_q. This approach allows us to refine in the capacitory terms previous results of about "singular" sets.

We are trying to give some interpretations to abstract (in some sense mystical) conditions of Ball's classes:
1) Why the condition $n-1<q$ is essential?;
2) When condition adj($D\varphi$ leads to invertibility?;
3) When this condition is necessary?;
4) Which structure has the "singular" set? "'

We show that a "topological" mapping of Sobolev classes $\varphi : \Omega \to \widetilde{\Omega}$, $J(x,\varphi) > 0$ for almost all $x \in \Omega$, belongs to $A^{+}_{q,r}(\Omega)$ if and only if φ possesses the Luzin N-property and for any function $f \in L^1_\infty(\widetilde{\Omega})$ (a bounded gradient of a stress tensor) the composition $\varphi^*(f) = f \circ \varphi \in L^1_q(\Omega)$ and

$$\|f \mid L^1_1(\widetilde{\Omega})\| \leq \|\operatorname{adj} D\varphi | L_r(\Omega)\| \cdot \|\varphi^*(f) \mid L^1_q(\Omega)\|,$$

where $r = q/(q-1)$.

This inequality states that the second Ball's condition $\operatorname{adj} D\varphi \in L_r(\Omega)$, $r = q/(q-1)$ is equivalent to the boundedness of the composition operator

$$\left(\varphi^{-1}\right)^* : L_q^1(\Omega) \to L_1^1(\widetilde{\Omega}). \tag{1}$$

generated by the inverse topological mapping $\varphi^{-1} : \widetilde{\Omega} \to \Omega$.
Remark The mapping φ is a homeomorphism from $\Omega \setminus S$ into $\widetilde{\Omega}$.

We study a special case of the variational problem

$$I(\varphi, \Omega) := \int_\Omega W(x, \varphi(x), D\varphi(x))\, dx,$$

for $W(x, \varphi(x), D\varphi(x) := |D\varphi(x)|^q$ i.e for $||\varphi||^q_{L^1_p}$.
All homeomorphisms $\varphi \in L^1_{1,loc}$ that we study here are mapping with finite distortion.
The mapping φ is called the mapping of finite distortion if $|D\varphi(z)| = 0$ for almost all $x \in Z = \{z \in \Omega : J(x, \varphi) = 0\}$. Here $J(x, \varphi)$ is Jacobian. (S. K. Vodop'yanov, V. M. Gol'dshtein, Yu. G. Reshetnyak, Uspekhi Mat. Nauk, 34 (1979).)
Mapping of Ball's classes have this property because $J(x, \varphi) > 0$ a.e..

Some basic concepts of the geometric theory of composition operators.
The main objects are topological mappings which generate bounded composition operators on Sobolev spaces by the composition rule. This theory is in some sense a "generalization" of the theory of quasiconformal mappings, but it is oriented to applications. This theory is motivated by the Reshetnyak's problem (1968) on geometric characterizations of isomorphisms φ^* of seminormed Sobolev spaces $L^1_n(\Omega)$ and $L^1_n(\widetilde{\Omega})$ generated by the composition rule $\varphi^*(f) = f \circ \varphi$). I do not refer here a history. The condition of "isomorphism" is very restricted for applications: for $p = n$ it is quasiconformal homeomorphism, for $p \neq n$ it is locally bi-Lipschitz homeomorphisms with uniformly restricted from both sides Lipschitz constant.

Sobolev homeomorphisms not necessarily induce isomorphism of corresponding Sobolev spaces. This is one of the main differences with scales of smooth functions and algebras of functions.
If Ω is an open subset of $\mathbb{R}^n$, the Sobolev space $W^1_p(\Omega)$, $1 \leq p \leq \infty$, is defined as a Banach space of locally integrable weakly differentiable functions $f : \Omega \to \mathbb{R}$ equipped with the following norm:

$$\|f \mid W^1_p(\Omega)\| = \|f \mid L_p(\Omega)\| + \|\nabla f \mid L_p(\Omega)\|.$$

The homogeneous seminormed Sobolev space $L^1_p(\Omega)$, $1 \leq p \leq \infty$, is defined as a space of locally integrable weakly differentiable functions $f : \Omega \to \mathbb{R}$ equipped with the following seminorm:

$$\|f \mid L^1_p(\Omega)\| = \|\nabla f \mid L_p(\Omega)\|.$$

Sobolev spaces are usually defined as Banach spaces of equivalence classes. To clarify the notion of equivalence classes we use the nonlinear p-capacity associated with Sobolev spaces. With the help of the p-capacity the Lebesgue differentiation theorem was refined for Sobolev spaces.

Suppose Ω is an open bounded set in $\mathbb{R}^n$ and $F \subset \Omega$ is a compact set. The p-capacity of F with respect to Ω is defined by

$$\operatorname{cap}_p(F;\Omega) = \inf\{\|\nabla f | L_p(\Omega)\|^p\},$$

where the infimum is taken over all functions $f \in C_0(\Omega) \cap L_p^1(\Omega)$ such that $f \geq 1$ on F.

By a usual limiting process this notion can be extended to any Borel sets. We are interested to sets of p-capacity zero. It is correct description because p-capacity is an outer measure. We omit technical details, but a better understanding connection between this outer measure and Hausdorf dimension is reasonable: If $\operatorname{cap}_p(F;\Omega)$ then any Hausdorff measure $\mu_\alpha(E) = 0$ for any $\alpha > n - p$.

It permits to refine the notion of Sobolev functions. Let a function $f \in L^1_p(\Omega)$. Then the refined function

$$\tilde{f}(x) = \lim_{r \to 0} \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} f(y)\, dy$$

is defined quasieverywhere i. e. up to a set of p-capacity zero and it is absolutely continuous on almost all lines. Recall that a function $\tilde{f}$ is termed quasicontinuous if for any $\varepsilon > 0$ there is an open set U_ε such that the p-capacity of U_ε is less than ε and on the set $\Omega \setminus U_\varepsilon$ the function $\tilde{f}$ is continuous. In what follows we will use the quasicontinuous (refined) functions only (reinterpretation of V.Maz'ya and V.Havin paper 1973).

A homeomorphism $\varphi : \Omega \to \widetilde{\Omega}$ induces a bounded composition operator

$$\varphi^* : L^1_p(\widetilde{\Omega}) \to L^1_q(\Omega), \;\; 1 \leq q \leq p \leq \infty,$$

by the composition rule $\varphi^*(f) = f \circ \varphi$, if for any function $f \in L^1_p(\widetilde{\Omega})$, the composition $\varphi^*(f) \in L^1_q(\Omega)$ is defined quasi-everywhere in Ω and there exists a constant $C_{p,q}(\Omega) < \infty$ such that

$$\|\varphi^*(f) \mid L^1_q(\Omega)\| \leq C_{p,q}(\Omega)\|f \mid L^1_p(\widetilde{\Omega})\|.$$

What is a minimal value of $C(p,q)$?
The p-distortion of a mapping φ at a point $x \in \Omega$ is defined as

$$K_p(x) = \inf\{k(x) : |D\varphi(x)| \leq k(x)|J(x,\varphi)|^{\frac{1}{p}},\ x \in \Omega\}.$$

If $p = n$ we have the usual conformal dilatation and in the case $p \neq n$ (Ghering 1962) the usual quasiconformal dilatation.
Its integral version is the following

$$K_{p,q}(\varphi;\Omega) = \|K_p \mid L_\kappa(\Omega)\| < \infty,$$

where $1/q - 1/p = 1/\kappa$ ($\kappa = \infty$, if $p = q$) (A.Ukhlov (1993), G. Gurov 1995).

A homeomorphism $\varphi : \Omega \to \widetilde{\Omega}$ is called a weak (p,q)-quasiconformal mapping, if $\varphi \in L^1_q(\Omega)$, has finite distortion, and

$$K_{p,q}(\varphi;\Omega) = \|K_p \mid L_\kappa(\Omega)\| < \infty,\ \ 1/\kappa = 1/q - 1/p\ (\kappa = \infty \text{ if } p = q),$$

Applications of weak (p,q)-quasiconformal mappings to embedding theorem and spectral theory were discussed in my previous talks. Here we discuss applications to Ball's classes.

Theorem (Ukhlov. 1993,2002) A homeomorphism $\varphi : \Omega \to \widetilde{\Omega}$ between two domains Ω and $\widetilde{\Omega}$ induces a bounded composition operator

$$\varphi^* : L^1_p(\widetilde{\Omega}) \to L^1_q(\Omega), \;\; 1 \leq q \leq p \leq \infty,$$

if and only if φ is a Sobolev mapping of the class $L^1_q(\Omega; \widetilde{\Omega})$, has finite distortion, posseses Luzin N-property and

$$K_{p,q}(\varphi; \Omega) = \| K_p \mid L_\kappa(\Omega) \| < \infty,$$

where $1/q - 1/p = 1/\kappa$ ($\kappa = \infty$, if $p = q$).

Let $\varphi : \Omega \to \widetilde{\Omega}$ be a topological mapping of Sobolev class. We proved in 2010 that $\varphi \in L^1_q(\Omega)$ if and only if the corresponding composition operator ("change of variables")

$$\varphi^* : L^1_\infty(\widetilde{\Omega}) \to L^1_q(\Omega)$$

is bounded. It means that the first Ball's condition $\varphi \in L^1_p(\Omega)$ has a reinterpretation in terms of composition operators. Recall that L^1_∞ is a space of all Lipschitz functions.

About the second Ball's condition. The following inequality was claimed before:

$$\|f \mid L_1^1(\widetilde{\Omega})\| \leq \| \operatorname{adj} D\varphi | L_r(\Omega)\| \cdot \|\varphi^*(f) \mid L_q^1(\Omega)\|,$$

where $r = q/(q-1)$.
It leads to conjecture that the inverse homeomorphism induce the composition operator from $L_q^1(\tilde{\Omega})$ to $L_1^1(\Omega)$. It is correct for $p > n$ and almost correct for other p

This inequality leads to corresponding capacity estimates:

$$\operatorname{cap}_1(\varphi(E);\widetilde{\Omega}) \leq \|\operatorname{adj} D\varphi | L_r(\Omega)\| \cdot \operatorname{cap}_q^{1/q}(E;\Omega)$$

for any q-capacity measurable set $E \subset \Omega$.
It means that that topological mappings of Ball's classes are absolutely continuous with respect to capacity and an image of a cavitation of q-capacity zero ("singular" set) in a body Ω has 1-capacity zero (also "singular" set) in $\widetilde{\Omega}$ and can not lead to the body breaking upon these deformations.

So, the cavitation can be characterized in capacitary terms. As a consequence we obtain characterization of cavitations in the terms of the Hausdorff measure: for every set $E \subset \Omega$ such that $\operatorname{cap}_q(E;\Omega) = 0$, $1 \leq p \leq n$, the Hausdorff measure $H^{n-1}(\varphi(E)) = 0$. This is an explanation why $q > n-1$ in the definition of Ball's classes.

The proposed approach allows us to "linearize" the nonlinear elasticity problems as corresponding problems for (linear and bounded) composition operators. It immediately leads to a possibility to use adequate functional analysis methods for the regularity problem (of inverse deformations).

The functional definition of Ball's classes.

We can define the Ball classes $A^+_{q,r}(\Omega)$ as class of Sobolev mappings $\varphi : \Omega \to \widetilde{\Omega}$, $\varphi \in L^1_q(\Omega)$, $J(x,\varphi) > 0$ for almost all $x \in \Omega$, such that

$$\|f \mid L^1_1(\widetilde{\Omega})\| \leq \|\operatorname{adj} D\varphi | L_r(\Omega)\| \cdot \|\varphi^*(f) \mid L^1_q(\Omega)\|,$$

for any $f \in L^1_\infty(\widetilde{\Omega})$, $q > n-1$, $r \geq q/(q-1)$.
For the case $n-1 < q < n$ it is necessary to add Luzin N-property into this definition.

Let us consider the case $n = 3$ and $q = n - 1 = 2$. Then the second condition is equivalent to the following condition

$$\left(\int\limits_{\widetilde{\Omega}} \frac{|D\varphi^{-1}(y)|^2}{|J(y, \varphi^{-1})|} \, dy \right)^{\frac{1}{2}} < \infty,$$

and it means that a ratio of deformations of line elements to deformations of (co-dimension 1) surface elements must be integrable, that seems to be natural from the point of view of mechanics of elastic bodies.

Remark Let $\lambda_3 \leq \lambda_2 \leq \lambda_1$ are singular values of $D\varphi^{-1}(y)$ then $\frac{|D\varphi^{-1}(y)|^2}{|J(y,\varphi^{-1})|} = \frac{\lambda_1}{\lambda_2 \lambda_3}$

Mappings of finite integral inner distortion allow us to prove existence of inverse composition operators in appropriate Sobolev spaces and its regularity properties. The inner distortion gives a geometric interpretation of the second condition about $adj(D(\varphi)$
In the paper (T. Iwaniec, J. Onninen, Z. Zhu, Deformations of bi-conformal energy and a new characterization of quasiconformality 2020) were considered bi-Sobolev mappings $\varphi : \Omega \to \widetilde{\Omega}$ such that $\varphi \in L^1_n(\Omega)$ and $\varphi^{-1} \in L^1_n(\widetilde{\Omega})$ in connections with the the non-linear elasticity problems. Note that these classes coincide with classes of weak $(n, 1)$-quasiconformal mappings.

Using the theory of composition operators we obtain the following volume distortion estimates for mappings of Ball's classes): *Let* $\varphi:\Omega\to\widetilde{\Omega}$ *belongs to* $A^+_{q,q'}(\Omega;\widetilde{\Omega})$, *then the following inequality*

$$|\varphi(A)|^{1-\frac{1}{n}} \leq \left(\frac{1}{|A|}\int\limits_A |\operatorname{adj} D\varphi(x)|^{\frac{q}{q-1}}\ dx\right)^{1-\frac{1}{q}} |A|^{1-\frac{1}{n}}, \quad n-1<q\leq n,$$

holds for any measurable set $A\subset\Omega$.
For classes $A^+_{q,r}(\Omega;\widetilde{\Omega})$ estimates of this type also exists but it is not a linear deformation estimates. It means that the restriction $r\geq\frac{q}{q-1}$ is not occasional.

By capacitary interpretation of this inequality can be concluded that the Ball's classes are absolutely continuous with respect to capacity and an image of cavitation of q-capacity zero ("singular" set) in a body Ω has 1-capacity zero (also "singular" set) in $\widetilde{\Omega}$ and can not lead to the body breaking upon these deformations. So, the cavitation can be characterized in capacitary terms. As a consequence we obtain characterization of cavitations in terms of the Hausdorff measure: for every set $E \subset \Omega$ such that $\operatorname{cap}_q(E;\Omega) = 0$, $1 \leq q \leq n$, the Hausdorff measure $H^{n-1}(\varphi(E)) = 0$.

Let $\varphi : \Omega \to \widetilde{\Omega}$ be a mapping of finite distortion of the class $W^1_{1,\mathrm{loc}}(\Omega)$. We define the inner q-distortion of φ at a point x as

$$K^I_q(x,\varphi) = \begin{cases} \frac{|J(x,\varphi)|^{\frac{1}{q}}}{l(D\varphi(x))}, & J(x,\varphi) \neq 0, \\ 0, & J(x,\varphi) = 0. \end{cases}$$

Here $l(D\varphi(x))$ is a minimal singular value of $D\varphi(x)$ i.e. $\min\limits_{h=1} |D\varphi(x)\cdot h|$

Its global integral version we call an inner (p,q)-distortion, $1 \le q \le p \le \infty$:

$$K^I_{p,q}(\Omega) = \|K^I_q(\varphi) \mid L_\kappa(\Omega)\|, \ 1/\kappa = 1/q - 1/p, \ (\kappa = \infty, \text{ if } p = q).$$

Remark Let $\varphi \in L^1_p(\Omega)$, $q = 1$ and $J(x,\varphi) > 0$. Using $(D\varphi(x))^{-1} = J^{-1}(x,\varphi)\,\mathrm{adj}\,D\varphi(x)$ and $\min\limits_{h=1} |D\varphi(x)\cdot h| = \left(\max\limits_{h=1} |(D\varphi(x))^{-1}\cdot h|\right)^{-1}$ we obtain

$$\left(K^I_{p,1}(\Omega)\right)^{\frac{p}{p-1}} = \int\limits_\Omega \left(\frac{|J(x,\varphi)|}{l(D\varphi(x))}\right)^{\frac{p}{p-1}} dx = \int\limits_\Omega |\mathrm{adj}\, D\varphi(x)|^{\frac{p}{p-1}} \ dx < \infty$$

Next theorems explain our generalization of the Ball's classes.
Theorem Let $\varphi : \Omega \to \widetilde{\Omega}$ be a homeomorphism of finite distortion that belongs to $L^1_p(\Omega)$. Then the inverse mapping $\varphi^{-1} : \widetilde{\Omega} \to \Omega$ generates a bounded composition operator

$$\left(\varphi^{-1}\right)^* : L^1_p(\Omega) \to L^1_p(\widetilde{\Omega}),\ 1 \leq p < \infty, \tag{2}$$

if and only if $\varphi^{-1} \in L^1_p(\widetilde{\Omega})$, possesses the Luzin N^{-1}-property and

$$K^I_{p,p}(\Omega) = \operatorname{esssup}_{x\in\Omega} \left(\frac{|J(x,\varphi)|}{I(D\varphi(x))^p} \right)^{\frac{1}{p}} < \infty.$$

Mappings of Ball's classes have finite distortion because $J(x,\varphi) > 0$ a.e.

A condenser in a domain $\Omega \subset \mathbb{R}^n$ is the pair (F_0, F_1) of connected disjoint closed relatively to Ω sets $F_0, F_1 \subset \Omega$. A continuous function $f \in L^1_p(\Omega)$ is called an admissible function for the condenser (F_0, F_1), if the set $F_i \cap \Omega$ is contained in some connected component of the set $\operatorname{Int}\{x | f(x) = i\}$, $i = 0, 1$. We call as the p-capacity of the condenser (F_0, F_1) relatively to domain Ω the following quantity:

$$\operatorname{cap}_p(F_0, F_1; \Omega) = \inf \|f | L^1_p(\Omega)\|^p.$$

Here the greatest lower bound is taken over all functions admissible for the condenser $(F_0, F_1) \subset \Omega$.

Theorem A homeomorphism $\varphi:\Omega\to\widetilde{\Omega}$ between two domains Ω and $\widetilde{\Omega}$ induces a bounded composition operator

$$\varphi^*: L^1_p(\widetilde{\Omega}) \to L^1_q(\Omega),\ \ 1\leq q\leq p\leq\infty,$$

if and only if $\varphi\in W^1_{1,\mathrm{loc}}(\Omega)$, has finite distortion, possesses Luzin N-property and

$$K_{p,q}(\varphi;\Omega)=\|K_p \mid L_\kappa(\Omega)\|<\infty,$$

where $1/q-1/p=1/\kappa$ ($\kappa=\infty$, if $p=q$).

Theorem Let a homeomorphism of finite distortion $\varphi : \Omega \to \widetilde{\Omega}$ belong to $L^1_p(\Omega)$, $1 < p < \infty$, possess the Luzin N-property and such that

$$K^I_{p,q}(\Omega) = \| K^I_q(\varphi) \mid L_\kappa(\Omega) \| < \infty, \; 1 < q < p < \infty,$$

where $1/\kappa = 1/q - 1/p$. Then for every condenser $(F_0, F_1) \subset \Omega$ the inequality

$$\operatorname{cap}_q^{\frac{1}{q}}(\varphi(F_0), \varphi(F_1); \widetilde{\Omega}) \leq K^I_{p,q}(\Omega) \operatorname{cap}_p^{\frac{1}{p}}(F_0, F_1; \Omega)$$

Theorem Let a homeomorphism of finite distortion $\varphi : \Omega \to \widetilde{\Omega}$ belong to $L^1_p(\Omega)$, $1 < p < \infty$, possess the Luzin N-property and such that

$$K^I_{p,q}(\Omega) = \| K^I_q(\varphi) \mid L_\kappa(\Omega)\| < \infty,\ 1 < q < p < \infty,$$

where $1/\kappa = 1/q - 1/p$. Then for every set $E \subset \Omega$ such that $\operatorname{cap}_p(E;\Omega) = 0$, $1 \leq p \leq n$, the Hausdorff measure $H^\alpha(\varphi(E)) = 0$ for any $\alpha > n - q$.

Some consequences for mappings φ of Ball's classes $A^+_{q,r}(\Omega)$. For the direct mapping φ the singular set has the q-capacity zero and has the α-Hausdorff measure zero for any $\alpha > n - q$.
Inverse mapping induce the composition operator

$$\left(\varphi^{-1}\right)^* : L^1_q(\Omega) \to L^1_1(\widetilde{\Omega}). \tag{3}$$

Hence $\widetilde{\Omega} \setminus \varphi(\Omega)$ has 1-capacity zero and has α-Hausdorff measure zero for any $\alpha > n - 1$ and can not produce cracks.

Vladimir Gol'dshtein and Alexander Ukhlov.Composition operators on Sobolev spaces and Ball's classes, arXiv:1905.00736.

J. M. Ball, Convexity condition and existence theorems in nonlinear elasticity, Arch. Rational Mech. Anal., 63 (1976), 337–403.ctions and the interpenetration of matter, Proc. Roy. Soc. Edinburgh Sect. A, 88 (1981), 315–328.

V. Gol'dshtein, A. Ukhlov, About homeomorphisms that induce composition operators on Sobolev spaces, Complex Var. Elliptic Equ., 55 (2010), 833–845.

P. Hajlasz, P.Koskela, Formation of cracks under deformations with finite energy, Calc. of Var., 19 (2004), 221-227.

THANKS